

# Conformal coordinates for a constant density star

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It is well known that the interior of a constant density spherical star is conformally flat. In this paper we obtain the coordinate system in which the conformal flatness of the metric manifests itself. In a similar way, we also construct such coordinates for Robertson Walker metric.

## I. INTRODUCTION

The constant density, spherically symmetric, perfect fluid solution to Einstein's equations was given in 1916 by Schwarzschild[1]. Despite being a stalwart of introductory General Relativity, the source of much of the knowledge we have about the Schwarzschild star interior solution is shrouded in obscurity. When he was writing about it in 1971, Buchdahl[2] decried the fact that no standard texts available to him even mentioned that the metric for this geometry is conformally flat. To our knowledge, the first documented record concerning this fact was published only a few years earlier by Shepley and Taub[3], in a work which Buchdahl actually did not cite. In fact, both of these papers seem to have been remarkably little read, or at least referred to, in the, almost forty, intervening years. Like many others from around that time, Shepley and Taub apparently came across the conformal flatness of the Schwarzschild interior solution in the course of establishing a different, somewhat more general, geometric result – including its unique properties among perfect fluid matter sources[2, 4]. For his part, Buchdahl wrote down coordinates in which the conformal flatness was manifest, but he neither established their existence specifically nor discussed their properties in any detail. Instead he concentrated on describing ensuing optical properties, for the systematic study of which he is justly held in high regard.

In this paper, we fill in specific gaps in Buchdahl's discussion. We find constructively the flat coordinates in which the conformal flatness is manifest and we discuss their domain of applicability and related properties in detail. Forming a backdrop to our presentation is the need to show a modern application (which will be published elsewhere) of calculating the self force of a static electric charge placed inside/outside a Schwarzschild star. Just as importantly, we also wish to prevent the earlier awareness [2] of this conformal flatness and its underlying implications from fading into oblivion.

The layout of the paper is as follows. We first demonstrate the conformal flatness of the spatial part of the metric by relating it to the metric on the 3-sphere. Then we find the family of conformal factors and related coordinates in which the conformal flatness of the 4-geometry is manifest, and discuss their domains of applicability. We then briefly address the implications of this work for the problem of static electromagnetic sources in the interior Schwarzschild star. Finally, we mimic our procedure to indicate the conformal coordinates for another conformally flat metric (Robertson-Walker).

## II. PREAMBLE

Consider the metric of a spherically symmetric, constant density star of mass  $M$  and radius  $R_s$  as given by Schwarzschild[1]. The exterior of the star is just the Schwarzschild metric (where the subscript refers to the Schwarzschild vacuum solution):

$$ds_{Sv}^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (1)$$

We preserve spherical symmetry throughout, and so denote the 2-sphere by the metric

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (2)$$

The star interior has the following metric (subscript refers to the **S**chwarzschild **s**tar):

$$ds_{Ss}^2 = -e^{2\Phi(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (3)$$

where  $\exp[\Phi(r)] = \frac{3}{2}\sqrt{1 - 2M/R_s} - \frac{1}{2}\sqrt{1 - 2Mr^2/R_s^3}$  and  $m(r) = M r^3/R_s^3$ . The quantity  $M/R_s$  is bounded by 4/9. For the extreme density case, when  $M/R_s = 4/9$ , the center  $r = 0$ , develops a singularity.

The Weyl tensor for this metric evaluates to zero, indicating that the metric is conformally flat. This implies that, about any point in the interior of the star, locally (for a finite region) there exist coordinates in which the metric would be Minkowski metric up to a conformal factor, that is  $g_{\mu\nu} = \Omega^2(x_\mu) \eta_{\mu\nu}$ . In particular, about  $r = 0$ , since we expect the spherical symmetry to be preserved, there should exist coordinates  $\{T(t, r), R(t, r), \theta, \phi\}$  such that (3) can be rewritten as (subscript fs refers to flat space):

$$ds_{\text{ss}}^2 = \Omega^2(t, r) [-dT^2 + dR^2 + R^2 d\Omega^2] = \Omega^2(t, r) ds_{\text{fs}}^2. \quad (4)$$

This coordinate transformation preserves spherical symmetry (it does not touch the angular variables). In this paper, we find the coordinates  $\{T(t, r), R(t, r)\}$  and the conformal factor  $\Omega(t, r)$ , which we refer to collectively as the conformal transformation.

In section IV, we modify the metric (3) by using a coordinate transformation to express  $ds_{\text{ss}}^2$  as  $\tilde{\omega}^2 d\tilde{s}^2$ , where  $d\tilde{s}^2$  is spatially flat. In section V, we work with the metric  $d\tilde{s}^2$  and find conformally flat coordinates for the extreme density star. We also rewrite the metric  $d\tilde{s}^2$  in a desired form (presumably used by Buchdahl) which, in Appendix A, we express in terms of a conformal factor  $\tilde{\Omega}$  and conformal coordinates  $\{T, R\}$ , before solving the differential equations they satisfy. In section VI, we similarly obtain the conformal factor  $\tilde{\Omega}$ , by demanding that the Riemann tensor of a flat metric vanishes. In section VII, we use the results from sections V, VI and appendix A to explicitly construct two sets of conformal coordinates. In section VIII, we look into a small application of these conformal coordinates in electrostatics. In section IX, we examine conformal coordinates for the Robertson-Walker metric (which is also conformally flat) by following the steps in section V through section VII. Finally, in appendix B, we find the conformal freedom relating (spherically symmetric) flat space to itself (which we use in section VIA).

### III. RECOGNIZING BUCHDAHL'S CONTRIBUTION

In his 1971 paper, Buchdahl lists three important results:

- By using a condition on the Weyl tensor ( $C_{klmn} = 0$ ), and by solving an appropriate Einstein equation for the perfect fluid, he established that the Schwarzschild interior solution represented the only static conformally flat distribution of fluid with nonnegative pressure and density.
- By relating the Schwarzschild star metric to the conformally transformed flat metric, he wrote down the coordinate transformation between the canonical coordinates for the Schwarzschild star and spherical polar coordinates in flat space. Interestingly for us, he gave virtually no details about his result, but he did include an equation for the flat space orbit of a point on the surface of the star.
- By introducing (conformally related) coordinates on the 3-sphere, he obtained rather directly an expression for the optical point characteristic — the time taken by light to propagate between two spatially distinct points. Throughout, he makes no explicit reference to the fact that the coordinates being introduced reside on the 3-sphere, nor to the fact that spatial slices of the Schwarzschild star are themselves scaled 3-spheres.

In this work we extend Buchdahl's analysis in several significant ways:

- We find the appropriate conformal factor by solving directly  $R_{klmn} = 0$  for the Riemann tensor of the conformally transformed Schwarzschild interior solution, rather than using the Weyl tensor to find the stellar properties. (section VI)
- We explicitly exhibit the conformal relation to the 3-sphere, and use it to find the coordinate transformation to the flat coordinates. (section V and Appendix A)
- We find the orbit of an arbitrary interior point, and use it to discuss the electromagnetic problem of a static point charge inside the Schwarzschild star, rather than the problem of light propagation. (section VIII)
- We fully characterize the parameter freedom in the choice of the conformal factor (and the different domains which arise from it), and demonstrate its 1–1 relationship with the conformal freedom to transform flat space into itself. (section VII)
- To do this, we have characterized the flat space conformal freedom explicitly. (Appendix B)
- We demonstrate the related properties of the Robertson-Walker metrics. (section IX)

#### IV. CONFORMAL PREPARATION

We first rewrite the interior metric (3) as

$$ds_{\text{Ss}}^2 = - \left[ a - \frac{1}{2} \sqrt{1 - \alpha r^2} \right]^2 dt^2 + \frac{1}{(1 - \alpha r^2)} dr^2 + r^2 d\Omega^2, \quad (5)$$

where  $a \equiv \frac{3}{2} \sqrt{1 - 2M/R_s}$  and  $\alpha \equiv 2M/R_s^3$ :  $a$  can take values in the range  $(\frac{1}{2}, \frac{3}{2})$ . For the extreme density case,  $a = 1/2$ . Next, we relabel the coordinates  $\sqrt{\alpha}r \rightarrow \hat{r}$  and  $\sqrt{\alpha}t \rightarrow \hat{t}$ . These new coordinates are dimensionless:

$$\alpha ds_{\text{Ss}}^2 = - \left[ a - \frac{1}{2} \sqrt{1 - \hat{r}^2} \right]^2 d\hat{t}^2 + \frac{1}{(1 - \hat{r}^2)} d\hat{r}^2 + \hat{r}^2 d\Omega^2. \quad (6)$$

The spatial 3-metric corresponds to a 3-sphere, as can be recognized by writing  $\hat{r} = \sin \eta$

$$ds_{\text{S3}}^2 = \frac{1}{1 - \hat{r}^2} d\hat{r}^2 + \hat{r}^2 d\Omega^2 \equiv d\eta^2 + \sin^2 \eta d\Omega^2.$$

Since the 3-sphere itself is conformally flat, we transform the coordinates to express this:

$$ds_{\text{S3}}^2 = \frac{d\hat{r}^2}{1 - \hat{r}^2} + \hat{r}^2 d\Omega^2 \rightarrow \hat{\omega}^2(\hat{r}) [d\gamma^2 + \gamma^2 d\Omega^2].$$

Because of the spherical symmetry, the above coordinate transformation will not disturb the angular coordinates  $\{\theta, \phi\}$ . The new radial coordinate  $\gamma$  depends only on  $\hat{r}$ , as does the conformal factor  $\hat{\omega}$ . Equating coefficients of the differentials in the two forms of the metric gives  $d\hat{r}/\sqrt{1 - \hat{r}^2} = \hat{\omega} d\gamma$  and  $\hat{r} = \hat{\omega} \gamma$ , which we solve by eliminating  $\hat{\omega}$ :

$$\frac{d\gamma}{d\hat{r}} = \frac{\gamma}{\hat{r} \sqrt{1 - \hat{r}^2}} \Rightarrow \gamma(\hat{r}) = \frac{2\hat{r}}{1 + \sqrt{1 - \hat{r}^2}}, \text{ and so} \quad (7)$$

$$\hat{\omega}(\hat{r}) = \frac{1 + \sqrt{1 - \hat{r}^2}}{2}, \text{ or } \hat{\omega}(\gamma) = \frac{4}{4 + \gamma^2}.$$

The interior metric in (6) is conformal to a spatially flat metric (as indicated in [4]):

$$\begin{aligned} \alpha ds_{\text{Ss}}^2 &= - \left[ a - \frac{1}{2} \sqrt{1 - \hat{r}^2} \right]^2 d\hat{t}^2 + \hat{\omega}^2(\gamma) [d\gamma^2 + \gamma^2 d\Omega^2], \\ &= - \left[ a - \frac{1}{2} \left( \frac{4 - \gamma^2}{4 + \gamma^2} \right) \right]^2 d\hat{t}^2 + \frac{16}{(4 + \gamma^2)^2} [d\gamma^2 + \gamma^2 d\Omega^2], \text{ and so :} \end{aligned}$$

$$\frac{(4 + \gamma^2)^2}{16} \alpha ds_{\text{Ss}}^2 = - \frac{1}{64} [4(2a - 1) + (2a + 1)\gamma^2]^2 d\hat{t}^2 + [d\gamma^2 + \gamma^2 d\Omega^2].$$

We now define  $\beta^2 \equiv (2a - 1)/(2a + 1)$ , which ranges between  $(0, 1)$ , being zero for the extreme density case. We again relabel coordinates, with  $\gamma/2 \rightarrow \tilde{r}$  and  $(2a + 1)\hat{t}/4 \rightarrow \tilde{t}$ . A conformal factor can be removed to make the metric take the following simple form:

$$ds_{\text{Ss}}^2 = \frac{4}{\alpha(1 + \tilde{r}^2)^2} [-(\beta^2 + \tilde{r}^2)^2 d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2] = \tilde{\omega}^2 d\tilde{s}^2, \quad (8)$$

where  $\tilde{\omega}^2 = 4/\alpha(1 + \tilde{r}^2)^2$ . Note that these new coordinates are dimensionless and the conformal factor in front of  $d\tilde{s}^2$  has the dimensions of  $\alpha^{-1}$ , which is  $[\text{L}]^2$ .

In this coordinate system, let the radius of the star be denoted as  $r_s$ . The coordinate  $\tilde{r}$  in metric (8) ranges from 0 to  $r_s$ . Tracing back the coordinate transformations performed so far, we can express  $r_s$  in terms of  $R_s$  as:

$$r_s = \frac{\sqrt{\frac{2M}{R_s}}}{1 + \sqrt{1 - \frac{2M}{R_s}}}. \quad (9)$$

No matter how big  $R_s$  is,  $r_s$  is always less than  $1/\sqrt{2}$ . For the extreme density star,  $r_s = 1/\sqrt{2}$ . We can express  $r_s$  and  $\beta$  purely in terms of each other as given below:

$$\beta^2 = \frac{(1 - 2r_s^2)}{2 - r_s^2}, \text{ and } r_s^2 = \frac{4(1 - 2\beta^2)}{2 - \beta^2}. \quad (10)$$

## V. CONFORMAL SOLUTION

The metric  $d\tilde{s}^2$  given in (8) can be expressed as manifestly conformally flat, with the help of a spherically symmetric conformal factor  $\tilde{\Omega}$  and new conformal coordinates  $\{T, R\}$ :

$$\begin{aligned} d\tilde{s}^2 &= -[\beta^2 + \tilde{r}^2]^2 d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \\ &= \tilde{\Omega}^2(\tilde{t}, \tilde{r}) [-dT^2 + dR^2 + R^2 d\Omega^2]. \end{aligned} \quad (11)$$

Consider first the extreme density case, when  $\beta = 0$ . It is straightforward to obtain:

$$T(\tilde{t}, \tilde{r}) = \tilde{t}, \quad R(\tilde{t}, \tilde{r}) = 1/\tilde{r}, \quad \Omega(\tilde{t}, \tilde{r}) = \tilde{r}^2. \quad (12)$$

The  $(\tilde{r}, \tilde{r})$  component of the Einstein tensor blows up at  $\tilde{r} = 0$ , exhibiting a singularity. Hence, performing a coordinate transformation at  $\tilde{r}=0$  is really pointless. We constrain our coordinate transformation so as to exclude  $\tilde{r} = 0$ . We let the coordinate transformation (12) be valid for  $\tilde{r} > r_0$ , for some small  $r_0 > 0$ , and for all values of  $\tilde{t}$ .

For  $\beta \neq 0$ , there is a further simplification which can be easily made. The metric

$$d\tilde{s}^2 = -(\beta^2 + \tilde{r}^2)^2 d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2, \quad (13)$$

is again conformal to a 4-metric with 3-spheres as spatial slices:

$$d\tilde{s}^2 = \bar{\omega}^2 \left[ -d\bar{t}^2 + d\bar{r}^2 + \sin^2(\bar{r}) d\Omega^2 \right] = \bar{\omega}^2 d\bar{s}^2, \quad \text{where} \quad (14)$$

$$\bar{t} = 2\beta\tilde{t}, \quad \bar{\omega} = (\beta^2 + \tilde{r}^2)/2\beta, \quad \text{and} \quad (15)$$

$$\bar{r} = 2 \arctan(\tilde{r}/\beta) \leq \mathcal{R}_s, \quad \text{and} \quad \mathcal{R}_s = 2 \arctan(r_s/\beta). \quad (16)$$

Then, as shown in appendix A, and where we chose the  $\mp$  to preserve time orientation:

$$d\tilde{s}^2 = \bar{\Omega}^2 [-dT^2 + dR^2 + R^2 d\Omega^2], \quad \text{where} \quad (17)$$

$$\bar{\Omega} = \bar{c} [\cos(\bar{r}) - \cos(\bar{t})], \quad (18)$$

$$R = \sin(\bar{r})/\bar{\Omega}, \quad \text{and} \quad (19)$$

$$T = \mp \sin(\bar{t})/\bar{\Omega}, \quad (20)$$

exemplify a transformation to flat space (with  $t_0=0$ ), where  $\bar{c}$  is selected so that  $\bar{\Omega} > 0$ .

## VI. CURVATURE EQUATIONS

From (17), it is clear that all the components of the Riemann tensor of the following metric should vanish:

$$ds_{\text{fs}}^2 = \bar{\Omega}(\bar{t}, \bar{r})^{-2} [-d\bar{t}^2 + d\bar{r}^2 + \sin^2(\bar{r}) d\Omega^2].$$

In this section, we shall solve for the conformal factor  $\bar{\Omega}(\bar{t}, \bar{r})$  for which the Riemann tensor vanishes. Evaluating the Riemann tensor reveals that there are 5 independent non vanishing components  $R_{\bar{t}\bar{r}\bar{t}\bar{r}}, R_{\bar{t}\bar{\theta}\bar{t}\bar{\theta}} = \sin^2 \theta R_{\bar{t}\bar{\phi}\bar{t}\bar{\phi}}, R_{\bar{t}\bar{\theta}\bar{r}\bar{\theta}} = \sin^2 \theta R_{\bar{t}\bar{\phi}\bar{r}\bar{\phi}}, R_{\bar{r}\bar{\theta}\bar{r}\bar{\theta}} = R_{\bar{r}\bar{\phi}\bar{r}\bar{\phi}}, R_{\bar{\theta}\bar{\phi}\bar{\theta}\bar{\phi}}$ . Equating each of these components to zero gives us 5 equations (in this section only, dot denotes a  $\bar{t}$ -derivative and prime an  $\bar{r}$ -derivative):

$$\bar{\Omega}^4 R_{\bar{t}\bar{r}\bar{t}\bar{r}} = \left[ \bar{\Omega}'^2 - \bar{\Omega} \bar{\Omega}'' - \dot{\bar{\Omega}}^2 + \bar{\Omega} \ddot{\bar{\Omega}} \right] = 0, \quad (i)$$

$$\bar{\Omega}^4 R_{\bar{t}\bar{\theta}\bar{t}\bar{\theta}} = \sin^2(\bar{r}) \left[ \bar{\Omega} \ddot{\bar{\Omega}} + \bar{\Omega}'^2 - \bar{\Omega} \bar{\Omega}' \cot(\bar{r}) - \dot{\bar{\Omega}}^2 \right] = 0, \quad (ii)$$

$$\bar{\Omega}^4 R_{\bar{t}\bar{\theta}\bar{r}\bar{\theta}} = \sin^2(\bar{r}) \bar{\Omega} \dot{\bar{\Omega}}' = 0, \quad (iii)$$

$$\bar{\Omega}^4 R_{\bar{r}\bar{\theta}\bar{r}\bar{\theta}} = \sin^2(\bar{r}) \left[ -\bar{\Omega}'^2 + \bar{\Omega} \bar{\Omega}' \cot(\bar{r}) + \bar{\Omega} \bar{\Omega}'' + \dot{\bar{\Omega}}^2 + \bar{\Omega}^2 \right] = 0, \quad (iv)$$

$$\bar{\Omega}^4 R_{\bar{\theta}\bar{\phi}\bar{\theta}\bar{\phi}} = \sin^4(\bar{r}) \sin^2 \theta \left[ \dot{\bar{\Omega}}^2 - \bar{\Omega}'^2 + 2\bar{\Omega} \bar{\Omega}' \cot(\bar{r}) + \bar{\Omega}^2 \right] = 0. \quad (v)$$

We now solve these five equations for  $\bar{\Omega}(\bar{t}, \bar{r})$  in a sequence of simple steps.

- We first note that these five equations are not all algebraically independent. Elimination of  $\dot{\bar{\Omega}}^2$  from (iv) and (v) gives

$$\bar{\Omega}' \cot(\bar{r}) - \bar{\Omega}'' = 0, \quad (\text{vi})$$

which also follows by eliminating  $\ddot{\bar{\Omega}}$  from (ii) and (i).

- It is similarly useful to simplify (i) and (iv), or equivalently (ii) and (v) to give:

$$\ddot{\bar{\Omega}} + \bar{\Omega}' \cot(\bar{r}) + \bar{\Omega} = 0. \quad (\text{vii})$$

Thus, to solve for  $\bar{\Omega}(\bar{t}, \bar{r})$ , it is sufficient to use just the equations (i),(iii),(vi),(vii).

- The simplest of these equations is (iii). Integrating (iii) gives  $\bar{\Omega}$  in terms of two arbitrary functions,  $F(\bar{t})$  and  $G(\bar{r})$ :

$$\bar{\Omega}(\bar{t}, \bar{r}) = F(\bar{t}) + G(\bar{r}). \quad (21)$$

- Substitution of this result into (vi) and then integrating gives:

$$G(\bar{r}) = A + B \cos(\bar{r}). \quad (22)$$

- Substitution of these results into (vii), and integrating, gives:

$$F(\bar{t}) = C \cos(\bar{t} - t_0) - A. \quad (23)$$

- Finally, substitution of all the results (21) through (23) into (i) determines

$$C = \pm B. \quad (24)$$

So, in the end we have, simply:

$$\bar{\Omega}(\bar{t}, \bar{r}) = B[\cos(\bar{r}) \pm \cos(\bar{t} - t_0)]. \quad (25)$$

The form of the time dependence,  $(\bar{t} - t_0)$ , signifies the time translation invariance of the metric (17). In principle, the  $\pm$  sign could be eliminated since a shift of  $t_0$  by  $\pm\pi$  would change the sign of the second cosine term. Keeping the  $\pm$  sign means  $t_0$  effectively lies in  $[0, \pi)$ . The scaling of  $B$  is really immaterial to the conformal transformation, though it is useful to retain, while the sign of  $B$  must match the sign of the expression inside the square brackets in (25), to ensure that  $\bar{\Omega}(\bar{t}, \bar{r})$  is always positive. This means that the family of conformal factors parameterized by  $B$  and  $t_0$  in (25) really splits into two subfamilies, depending on the sign of the square bracket in (25) or, equivalently, the sign of  $B$ . For each  $t_0$ , the members of each subfamily are bounded by (null) surfaces upon which  $\bar{\Omega}(\bar{t}, \bar{r}) = 0$ . These separate the domains in which  $B$  is positive from those in which  $B$  is negative. Taken together, the different domains from the two subfamilies are complementary, in that they then cover the entire spacetime.

### A. Flat space comparison

In appendix B, we study the properties of the most general spherically symmetric conformal transformations on flat space which can be obtained by a coordinate transformation. The results are summarized in (B2). We see there that the conformal factor  $H(T, R)$  also has two free parameters,  $D$  and  $T_0$ . A unique relationship exists between these two parameters and the changes they induce in the two parameters  $\{B, t_0\}$  in  $\bar{\Omega}(\bar{t}, \bar{r})$  as given in (25). We now find this relationship.

Consider the metric  $d\bar{s}^2$  in (11). Let us transform the conformal coordinates  $(T, R)$  to new set of coordinates  $(T', R')$  such that a conformal factor  $H(T, R)$  is pulled out as in shown in (B1):

$$d\bar{s}^2 = \bar{\Omega}^2 [-dT^2 + dR^2 + ..] = \bar{\Omega}^2 H^2 [-dT'^2 + dR'^2 + ..] \quad (26)$$

Since (25) gives the most general form of  $\bar{\Omega}$ , we know that, for any given  $\bar{\Omega}(\bar{t}, \bar{r})$  (fixed  $B, t_0$ ) and  $H(T, R)$  (fixed  $D, T_0$ ), the factor  $\bar{\Omega}H$  should be of the same form as the  $\bar{\Omega}$  given in (25), in which corresponding parameters  $B'$  and  $t'_0$  can be uniquely determined in terms of  $\{B, t_0\}$  and  $\{D, T_0\}$ . Thus, the product  $\bar{\Omega}H$  must take the form,

$$\bar{\Omega}H = B' [\cos(\bar{r}) \pm \cos(\bar{t} - t'_0)] \equiv \bar{\Omega}'. \quad (27)$$

It can be shown that

$$B' = \frac{(B^2 T_0^2 + 1)D}{B}, \text{ and } t'_0 = t_0 \pm 2 \arctan\left(\frac{1}{BT_0}\right). \quad (28)$$

Note that the parameter  $T'_0$  of (B2) has also been changed, as similarly occurs in (B4):

$$\delta T'_0 = \frac{B^2 T_0}{(B^2 T_0^2 + 1)D}. \quad (29)$$

Inverting, we can find  $\{D, T_0\}$ , for a given  $\{B, t_0\}$ , in terms of any chosen  $\{B', t'_0\}$ :

$$D = BB' \sin^2[(t'_0 - t_0)/2], \text{ and } T_0 = \pm \frac{\cot[(t'_0 - t_0)/2]}{B}. \quad (30)$$

Clearly, every member in the family of conformal factors  $\bar{\Omega}$  given in (25) can be obtained from just one of their representatives through conformal transformations as shown in appendix B. The degrees of freedom available for choosing a conformal factor of the form (25) entirely correspond to the degrees of freedom available in choosing a conformal factor  $H$  which still maintains a flat coordinate system, as shown in appendix B.

## VII. STELLAR COORDINATE DOMAINS

In this section, we consider two specific forms of  $\bar{\Omega}(\bar{t}, \bar{r})$  (one from each subfamily), by suitably choosing the  $\pm$  sign, and the parameters  $B$  and  $t_0$  in (25). We then construct the conformal coordinates  $R(\bar{t}, \bar{r})$  and  $T(\bar{t}, \bar{r})$  corresponding to each of these  $\bar{\Omega}$ . One of these coordinate transformations (part A) is well defined everywhere within a given range of  $\bar{t}$  around  $\bar{t} = t_0$ , while the other set of coordinates (part B) is never well defined at the origin  $\bar{r} = 0$ . For the coordinates in part B, the limit  $\beta \rightarrow 0$  will correspond to (12). The set of coordinates in part A does not have such a well defined limit. In part C, we provide the inverse coordinate transformations corresponding to the coordinate transformations of part A.

### A.

First, we consider  $\bar{\Omega}$  as given in (25) with  $B = 1$ , the +ve sign taken, and  $t_0 = 0$ :

$$\bar{\Omega}(\bar{t}, \bar{r}) = \cos(\bar{r}) + \cos(\bar{t}). \quad (31)$$

From (A3 III), we obtain  $R(\bar{t}, \bar{r})$  while from (20) or (A18) we take  $T(\bar{t}, \bar{r})$  and write:

$$R(\bar{t}, \bar{r}) = \frac{\sin(\bar{r})}{\bar{\Omega}(\bar{t}, \bar{r})}, \text{ and } T(\bar{t}, \bar{r}) = \frac{\sin(\bar{t})}{\bar{\Omega}}. \quad (32)$$

For each  $\bar{r}$ , the coordinate transformation is locally valid for  $\bar{t}$  in the range (see Fig 1):

$$-\pi + \bar{r} < \bar{t} < \pi - \bar{r}. \quad (33)$$

Nevertheless, this transformation maps region 1 of Fig 1 into the whole of flat space. Since  $\bar{r} \leq \mathcal{R}_s$ ,  $\pi - \mathcal{R}_s$  is a lower bound for  $\pi - \bar{r}$ . Hence, the range of unconditional validity inside the star is a strip of finite width in the  $t$ -coordinate:  $|\bar{t}| < \pi - \mathcal{R}_s$ .

## B.

Next, we choose parameters  $\{\pm, B, t_0\}$  in (25) to give the conformal transformation as:

$$\overline{\Omega}(\bar{t}, \bar{r}) = \cos(\bar{t}) - \cos(\bar{r}), \text{ and} \quad (34)$$

$$R(\bar{t}, \bar{r}) = \frac{\sin(\bar{r})}{\overline{\Omega}}, \quad T(\bar{t}, \bar{r}) = \frac{\sin(\bar{t})}{\overline{\Omega}}. \quad (35)$$

For each  $\bar{r}$ , the coordinate transformation (35) is valid for  $\bar{t}$  in the region  $-\bar{r} < \bar{t} < \bar{r}$  (see Fig 1). This transformation similarly maps region 2 in Fig 1 into the whole of flat space. Unlike in the previous case (A), there exists no nonzero lower bound for  $\bar{r}$ .

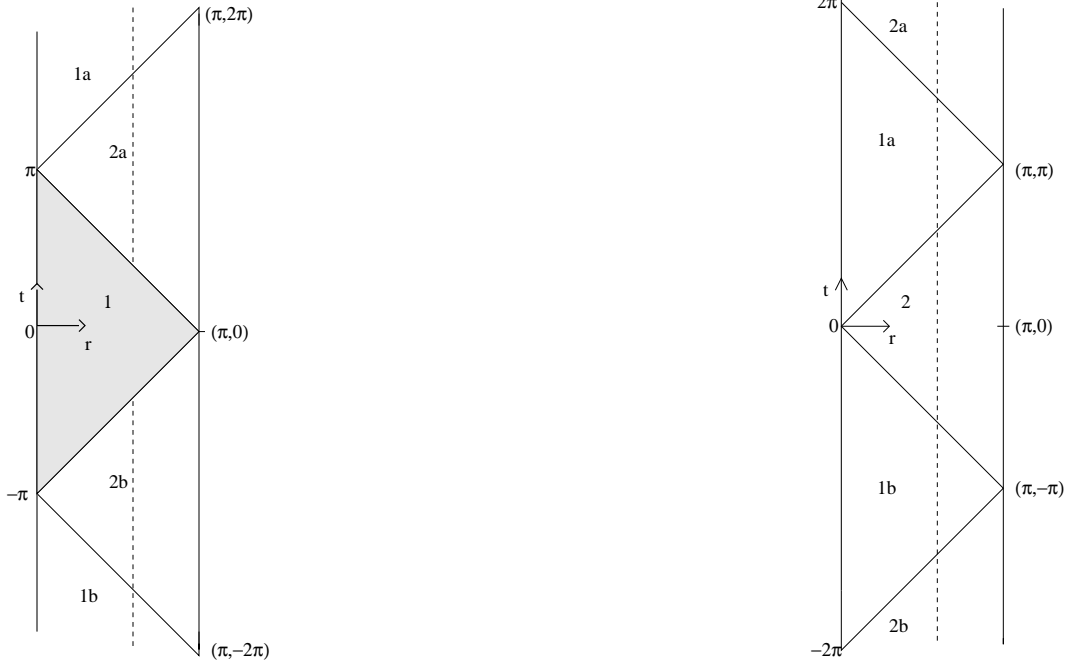


FIG. 1: On the left, the region mapped by  $\overline{\Omega}$  given in part A is shown shaded, and labeled 1. Regions 1a, 1b are also mapped by this transformation. To map regions 2a, 2b requires an overall sign change in (31). On the right, the regions mapped by  $\overline{\Omega}$  in part B are labeled 2, and 2a, 2b. Regions 1a, 1b require an overall sign change in (34).

At this point we restore the  $\beta$  dependence by reintroducing (15,16), along with the coordinates of (13). For  $\beta$  away from zero, the coordinate transformation (35) is not valid at the origin,  $\bar{r} = 0, \bar{t} = 0$ . Recall that in section V, for  $\beta = 0$ , we took the coordinate transformation (12) to apply only in the region  $\tilde{r} > r_0$ . Similarly, by restricting  $\tilde{r}$  to be greater than  $r_0$  when considering (15) and (16) in the coordinate transformation in (35), we have a nonzero lower bound for  $\tilde{r}$ , and thus, in the coordinates of (13):

$$-\frac{1}{\beta} \tan^{-1} \left( \frac{r_0}{\beta} \right) < \tilde{t} < \frac{1}{\beta} \tan^{-1} \left( \frac{r_0}{\beta} \right). \quad (36)$$

In the limit  $\beta \rightarrow 0$ , it can be shown that (35) reduces to (12) while the region of validity of the coordinate transformation (36) becomes  $|\tilde{t}| < \infty$ , exactly as in (12).

## C.

Since the conformal coordinates given in part A (32) have  $r = 0 \Leftrightarrow R = 0$  for all  $\{T, t\}$ , and are well behaved near  $r = 0$ , it is possible to obtain the inverse coordinate transformations  $r(T, R), t(T, R)$  there. In fact, we first invert a

more general expression of the coordinate relations before applying them to the specific case of interest. Thus, we consider:

$$\bar{\Omega}(\bar{t}, \bar{r}) = |B| [\epsilon_1 \cos(\bar{r}) + \epsilon_2 \cos(\bar{t})], \text{ with} \quad (37)$$

$$R(r, t) = \frac{\sin(\bar{r})}{\bar{\Omega}}, \text{ and} \quad T(\bar{t}, \bar{r}) = \frac{\epsilon_2 \sin(\bar{t})}{\bar{\Omega}}, \quad (38)$$

in which time orientation is preserved; the quantities  $\epsilon_1, \epsilon_2$  may independently be  $\pm 1$ , and the equations apply only in domains where  $\bar{\Omega} \geq 0$ . These can be inverted to give:

$$\tan(\bar{r}) = \frac{\epsilon_1 2|B|R}{B^2(T^2 - R^2) + 1}, \text{ and} \quad \tan(\bar{t}) = -\frac{2|B|T}{B^2(T^2 - R^2) - 1}. \quad (39)$$

In the domain of part A,  $\epsilon_1$  and  $\epsilon_2$  are both  $+1$ . It is interesting to note that the results in (39) are independent of  $\epsilon_2$ , so they apply unchanged for both signs indicated in (25).

### A Point to remember

The conformal transformations in (31,32,34,35) are not expressed in terms of the original Schwarzschild coordinates  $(\bar{t}, \bar{r})$  as given in (3); rather they are expressed in terms of the coordinates  $(\bar{t}, \bar{r})$  as given in (14). Tracing back the coordinate transformations performed in section IV and section V gives the relation between the coordinates in (3) and (14). To obtain the conformal factor and the conformal coordinates in terms of the original Schwarzschild coordinates in (3), we have to multiply  $\bar{\Omega}$  by  $\tilde{\omega}$  (see (8)) and by  $\bar{\omega}$  (see (14)), and replace  $\bar{r}$  and  $\bar{t}$  by the following functions:

$$\bar{t} = \sqrt{\alpha(a^2 - 1/4)}t, \text{ and} \quad \bar{r} = 2 \arctan \left( \frac{r}{1 + \sqrt{1 - \alpha r^2}} \sqrt{\frac{\alpha(2a + 1)}{(2a - 1)}} \right). \quad (40)$$

## VIII. STELLAR APPLICATION: ELECTROSTATICS

Since Maxwell's equations are conformally invariant, and since the Schwarzschild star metric in (5) is conformally flat, any electromagnetic problem inside the Schwarzschild star can be translated into an electromagnetic problem in Minkowski space. We use the coordinate system  $(\bar{t}, \bar{r}, \theta, \phi)$  and the Schwarzschild star metric as in (8) and (14), and for the flat metric, we shall use the coordinate system  $(T, R, \theta, \phi)$  as given in section VII. We consider a static point charge  $e$ , inside the Schwarzschild star at  $\bar{r} = \bar{r}_0$ . When we translate the problem into flat spacetime electrodynamics, the charge will not be static. We describe its motion by the function  $R = \Re(T, \bar{r}_0)$ , using (39), without reference to  $\bar{t}$ :

$$\Re = \frac{-\cos(\bar{r}_0) + \sqrt{1 + \sin^2(\bar{r}_0)T^2}}{\sin(\bar{r}_0)}. \quad (41)$$

The four velocity of the charge is given by

$$U^\mu = \frac{dT}{d\tau} \left( 1, \frac{d\Re}{dT}, 0, 0 \right),$$

where  $\tau$  is the proper time of the charge. From (41), we have

$$\frac{d\Re}{dT} = \frac{\sin(\bar{r}_0)T}{\sqrt{1 + \sin^2(\bar{r}_0)T^2}}. \quad (42)$$

Hence,  $U^\mu$  can be expressed as

$$U^\mu = \left( \sqrt{1 + \sin^2(\bar{r}_0)T^2}, \sin(\bar{r}_0)T, 0, 0 \right), \quad (43)$$



in which the normalization has fixed  $U^\mu U_\mu = -1$ . The proper time of the charge  $\tau$ , can be obtained directly from (43):

$$\frac{dT}{d\tau} = \sqrt{1 + \sin^2(\bar{r}_0)T^2} \quad \Rightarrow \quad \tau = \frac{1}{\sin(\bar{r}_0)} \ln \left[ \sin(\bar{r}_0)T + \sqrt{1 + \sin^2(\bar{r}_0)T^2} \right]. \quad (44)$$

The four acceleration  $a^\mu$  of the charge is given by:

$$a^\mu = \frac{dU^\mu}{d\tau} = \frac{dU^\mu}{dT} \frac{dT}{d\tau} = \left( \sin^2(\bar{r}_0)T, \sin(\bar{r}_0)\sqrt{1 + \sin^2(\bar{r}_0)T^2}, 0, 0 \right). \quad (45)$$

An interesting property of this acceleration is that its magnitude is a constant,  $a^\mu a_\mu = \sin^2(\bar{r}_0)$ . We end by concluding that the problem of electrostatics inside the Schwarzschild star corresponds to electrodynamics in a flat geometry with the current density  $J^\mu(T) = e\delta(R - \Re(T)) U^\mu(T)/\sqrt{1 + \sin^2(\bar{r}_0)T^2}$ .

## IX. ROBERTSON-WALKER METRIC

We finish with a brief discussion of the Robertson-Walker metric, which is also a conformally flat metric (Weyl tensor evaluates to zero). It often appears as a matter solution of the Einstein equations in contexts in which the Schwarzschild interior solution is also discussed, and it has similar conformal properties. We have:

$$ds_{\text{RW}}^2 = \eta^2(t) \left[ -dt^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]. \quad (46)$$

Here  $\eta(t)$  is the expansion factor of the universe and  $k = 0, \pm 1$ . For  $k = +1$  (spatially closed universe), the coordinate  $r$  ranges from 0 to 1. For  $k = 0$  or  $-1$  (spatially open), the coordinate  $r$  ranges from 0 to  $\infty$ .

Following the procedure of sections V through VII, we can find the conformal coordinates of this metric. For  $k = 0$ , these coordinates are by themselves conformal coordinates. For  $k = \pm 1$ , we can perform coordinate transformations as in section IV,

$$\tilde{r} = \frac{r}{1 + \sqrt{1 - kr^2}}, \quad \text{and} \quad \tilde{t} = t/2, \quad (47)$$

to obtain the metric in the form:

$$ds_{\text{RW}}^2 = \frac{4\eta^2(t)}{(1 + k\tilde{r}^2)^2} \left[ -(1 + k\tilde{r}^2)^2 d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \right]. \quad (48)$$

This is equivalent to (8) with  $\tilde{\omega} \equiv 2\eta(t)/(1 + k\tilde{r}^2)$ ,  $\beta = 1$  and  $0 \leq \tilde{r} \leq 1$ , for  $k = \pm 1$ . Now, following the steps of sections V and VI, we can obtain the conformal factor  $\tilde{\Omega}(\tilde{t}, \tilde{r})$  which makes the metric  $\tilde{\Omega}^{-2} d\tilde{s}^2$  flat.

For  $k = 1$ , we obtain, as previously,  $\tilde{\Omega} = \tilde{\omega}\tilde{\Omega}$ , with  $\tilde{\omega}$  as in (14) and  $\tilde{\Omega}$  as in (25), hence the same form for the conformal coordinates as in sections V and VII. The corresponding results for the Schwarzschild star thus apply exactly for the  $k = 1$  case of the RW metric.

For  $k = -1$ , note that  $g_{\tilde{t}\tilde{t}} = -(\beta^2 - \tilde{r}^2)$ , which changes the results from step (14). We will distinguish between the two families of  $\tilde{\Omega}(\tilde{t}, \tilde{r})$  which ensue, because of their rather different character relative to the result in (25). They are related by a  $\pm i\pi$  shift in  $t_0$ :

$$\tilde{\Omega}_1(\tilde{t}, \tilde{r}) = B [\cosh(\tilde{r}) + \cosh(\tilde{t} - t_0)], \quad (49)$$

$$\tilde{\Omega}_2(\tilde{t}, \tilde{r}) = B [-\cosh(\tilde{r}) + \cosh(\tilde{t} - t_0)]. \quad (50)$$

The time dependence is again of the form  $t - t_0$  due to the time translation invariance of the RW metric. With  $B = 1$  and  $t_0 = 0$  for  $\tilde{\Omega}_1$  given in (49), and by following steps as in appendix A, we obtain the following conformal coordinates:

$$R(\tilde{t}, \tilde{r}) = \frac{\sinh(\tilde{r})}{[\cosh(\tilde{r}) + \cosh(\tilde{t})]}, \quad \text{and} \quad T(\tilde{t}, \tilde{r}) = \frac{\sinh(\tilde{t})}{[\cosh(\tilde{r}) + \cosh(\tilde{t})]}. \quad (51)$$

Unlike the corresponding conformal coordinates for Schwarzschild star (section VII), these conformal coordinates cover the entire spacetime, since this  $\bar{\Omega}$  never vanishes.

By contrast, members of the family of conformal factors shown in (50) are not valid everywhere in the spacetime, and as in the case of Schwarzschild star, this family ( $\bar{\Omega}_2$ ), splits into two subfamilies, one with  $B > 0$  and the other with  $B < 0$ . For each  $t_0$ , their domains of validity are complementary, in that together, they cover the entire spacetime. Choosing  $B=1$  and  $t_0=0$  for  $\bar{\Omega}_2$ , we obtain the following conformal coordinates:

$$R(\bar{t}, \bar{r}) = \frac{\sinh(\bar{r})}{[-\cosh(\bar{r}) + \cosh(\bar{t})]}, \quad \text{and} \quad T(\bar{t}, \bar{r}) = \frac{\pm \sinh(\bar{t})}{[-\cosh(\bar{r}) + \cosh(\bar{t})]}. \quad (52)$$

Note: whereas (34) and (35) apply in the domain  $|\bar{t}| < \bar{r}$ , (52) applies in  $\bar{r} < |\bar{t}|$ .

## X. DISCUSSION AND SUMMARY

We have constructed conformal coordinate systems for two conformally flat geometries, the Schwarzschild interior solution and the Robertson-Walker metric. Their conformal flatness is manifest in these coordinates. Without involving the spherically symmetric 2-spheres, we show that these coordinate systems have two degrees of freedom, one corresponding to a scaling factor  $B$ , and the other corresponding to time translation  $t_0$ .

The Schwarzschild star metric (8) and the RW metric (48) for a closed universe ( $k=1$ ) require identical treatment for finding their conformal coordinates. By explicitly choosing  $t_0$  and  $B$ , we arrive at specific choices for these conformal coordinates, (31, 34). These coordinates cover only a part of the entire spacetime (see, for example (33)). Complementary regions of the spacetime (not covered by the chosen coordinates) are covered by other coordinate systems having the same  $t_0$  as (31, 34), but a negative  $B$ . We consider these complementary coordinate systems to belong to distinct subfamilies.

For the open universe ( $k=-1$ ) RW metric, we obtain two families of coordinate systems. A specific member from the first family is given by (51). This coordinate system is well behaved and covers the entire spacetime. Members of the second family do not individually cover the entire spacetime. As in the  $k=1$  case, they further split into two subfamilies, which are complementary in that together they cover the entire spacetime. A specific member from the second family is given by (52).

A useful application of these conformal coordinates is in electrodynamics, because Maxwell's equations are conformally invariant. In solving these equations, it suffices to use flat space Green's functions, which simplify calculations somewhat. We do have to expend some of the effort saved in order to find the appropriate current density  $J^\mu$ , which requires further analysis. In section VIII, we examine the orbit of a static point source, that becomes dynamical in the flat coordinates. For electromagnetic self-force calculations in conformally flat geometries, these conformal coordinates are proving useful.

## APPENDIX A: TRANSFORMATION FROM THE 3-SPHERE TO FLAT SPACE

We start from the 4-metric with spatial 3-sphere slices, with the form obtained in (14):

$$ds^2 = -dt^2 + dr^2 + \sin^2(r)d\Omega^2, \quad (A1)$$

and introduce flat coordinates  $T(t, r)$  and  $R(t, r)$  as in (4). Expressing the differentials of the new coordinates in terms of the old coordinates, we can write (A1) as:

$$ds^2 = -dt^2 + dr^2 + \sin^2(r)d\Omega^2 = \Omega^2(t, r) \left[ -(\dot{T}^2 - \dot{R}^2)dt^2 + (R'^2 - T'^2)dr^2 + 2(R'\dot{R} - T'\dot{T})dtdr + R^2d\Omega^2 \right], \quad (A2)$$

where a dot denotes the derivative with respect to  $t$  and a prime denotes the derivative with respect to  $r$ . Equating the coefficients of identical differentials on either sides gives a set of relations in which  $T$  does not appear undifferentiated:

$$\begin{aligned} 1 &= \Omega^2(\dot{T}^2 - \dot{R}^2), & I \\ \Omega^2(R'^2 - T'^2) &= 1, & II \\ \sin^2(r) &= \Omega^2 R^2, & III \\ \dot{T}T' &= \dot{R}R', \text{ and thus } & IV \\ \dot{T}^2 + T'^2 &= \dot{R}^2 + R'^2, & V \end{aligned} \quad (A3)$$

where the last result follows after eliminating  $\Omega$  from equations I and II. Note: these last two equations can be combined to give, by elementary algebra:

$$\begin{aligned}\dot{T} &= \mp R', \\ T' &= \mp \dot{R},\end{aligned}\tag{A4}$$

where only results compatible with I and II have been retained. Further differentiation and elimination gives the 2-D wave equation for both  $T$  and  $R$ , which because of the relations (A4), yields the solutions:

$$\begin{aligned}R(t, r) &= f(t - r) + g(t + r), \text{ and} \\ T(t, r) &= \pm [f(t - r) - g(t + r)].\end{aligned}\tag{A5}$$

Elimination of  $\Omega$  from I and III, and substitution of (A5) into the result, gives:

$$(f + g)^2 + 4\dot{f}\dot{g}\sin^2(r) = 0,\tag{A6}$$

while equations I and II together directly give:

$$-4\dot{f}\dot{g} = \Omega^{-2} = 4f'g',\tag{A7}$$

indicating that  $f'$  and  $g'$  must have everywhere the same sign.

By taking the  $t$  derivative of (A6) and eliminating  $\sin^2(r)$  from the result, we obtain an equation which can be rearranged to give:

$$\left(2\dot{f} - \frac{f\ddot{f}}{\dot{f}}\right) - \left(g\frac{\ddot{f}}{\dot{f}} + f\frac{\ddot{g}}{\dot{g}}\right) + \left(2\dot{g} - \frac{g\ddot{g}}{\dot{g}}\right) = 0.\tag{A8}$$

Futher differentiation, once by the argument of  $f$  and once by the argument of  $g$ , then rearranging to separate variables, gives (with  $\alpha$  a constant of separation):

$$\frac{1}{\dot{f}} \left( \frac{\ddot{f}}{\dot{f}} \right)' = -\frac{1}{\dot{g}} \left( \frac{\ddot{g}}{\dot{g}} \right)' = -\alpha,\tag{A9}$$

which can be solved completely. Integrating once gives (with integration constants  $a, c$ ):

$$\frac{\ddot{f}}{\dot{f}} = -\alpha(f + a), \text{ and } \frac{\ddot{g}}{\dot{g}} = \alpha(g + c).\tag{A10}$$

Substitution of these back into (A8) allows further separation of variables (with separation constant  $k$ ):

$$2\dot{f} - \frac{f\ddot{f}}{\dot{f}} - \alpha cf = -2\dot{g} + \frac{g\ddot{g}}{\dot{g}} - \alpha ag = -k,\tag{A11}$$

while further integration of (A10) gives (with new integration constants  $b, d$ ):

$$\dot{f} = -\frac{\alpha}{2} [(f + a)^2 + b^2], \text{ and } \dot{g} = \frac{\alpha}{2} [(g + c)^2 + d^2].\tag{A12}$$

Substitution of (A12) and (A10) back into (A11) allows us to conclude:

$$c = -a, \quad d = \pm b, \text{ and } k = \alpha(a^2 + b^2).\tag{A13}$$

Finally, further integration of (A12) yields (with additional integration constants  $e, h$ ):

$$f = -a + b \cot \frac{\alpha b}{2}(t - r + e), \text{ and } g = a - b \cot \frac{\alpha b}{2}(t + r + h),\tag{A14}$$

unless  $b = 0$  (relevant for the flat space treatment in appendix B), in which case:

$$f = -a + \frac{2}{\alpha(t - r + e)}, \text{ and } g = a - \frac{2}{\alpha(t + r + h)}.\tag{A15}$$

Up to this point nothing has depended on the metric coefficient multiplying the metric on the 2-sphere. Now, using the solution (A14) back in (A6) where that coefficient appears, and simplifying, gives us:

$$\alpha b \sin(r) = \pm \sin \alpha b \left( r + \frac{h-e}{2} \right), \quad (\text{A16})$$

from which we conclude (representing time translation invariance through the offset  $t_0$ ):

$$\alpha b = 1, \quad e = -t_0 - n\pi, \quad \text{and} \quad h = -t_0 + n\pi, \quad (\text{A17})$$

where  $n$  is an integer (+ve or -ve), even for the + sign in (A16), odd for the - sign. We can now solve for  $R$ ,  $T$  (after replacing  $a$  by  $\mp T_0/2$ ) and  $\Omega$ :

$$\begin{aligned} R(t, r) &= \frac{2}{\alpha} \frac{\sin(r)}{\cos(r) - (-1)^n \cos(t - t_0)}, \\ T(t, r) &= \frac{2}{\alpha} \frac{\pm \sin(t - t_0)}{\cos(r) - (-1)^n \cos(t - t_0)} + T_0, \\ \Omega(t, r) &= \frac{\alpha}{2} [\cos(r) - (-1)^n \cos(t - t_0)]. \end{aligned} \quad (\text{A18})$$

Note: we must choose  $\alpha$  so that  $R, \Omega > 0$ , we may, in principle, absorb the  $(-1)^n$  into a redefinition of  $t_0$ , and we would want the  $\pm$  sign to preserve time orientation.

## APPENDIX B: TRANSFORMATIONS WITHIN FLAT SPACE

We consider the flat space metric in spherical polar coordinates and find the most general, spherically symmetric, conformal transformation  $H$ , that maintains the flatness of this metric. In other words, we find  $H(T, R)$ , and the coordinate functions  $R'(T, R)$  and  $T'(T, R)$ , which satisfy the following equation:

$$[-dT^2 + dR^2 + R^2 d\Omega^2] = H^2 [-dT'^2 + dR'^2 + R'^2 d\Omega^2]. \quad (\text{B1})$$

By equating the components of the Riemann tensor of the above metric to zero, we obtain equations similar to those in section 3 {eq(i)-(v)}, with  $\sin(r) \rightarrow r$  and  $\Omega$  replaced by  $H$ . These equations can be solved in a way similar to that used in section VI, and the most general solution is found to be:

$$\begin{aligned} H(T, R) &= D [(T - T_0)^2 - R^2], \\ T' - T_0' &= \mp (T - T_0)/H, \quad \text{and} \\ R' &= R/H, \end{aligned} \quad (\text{B2})$$

where  $D$  and  $T_0$  are arbitrary parameters. Null infinity becomes the lightcone emerging from  $T_0'$ . The new coordinates  $\{R', T'\}$  and the conformal factor  $H$  are singular on the light cone emerging from the point  $R = 0, T = T_0$ . We refer to this light cone as  $L(T_0)$ .

The requirement that the conformal factor be positive definite ( $H > 0$ ), splits the family of conformal factors (parameterised by  $D$  and  $T_0$ ) naturally into two subfamilies, one with  $D > 0$  and the other with  $D < 0$ . The absolute magnitude of  $D$  corresponds to a relatively trivial scaling factor which can be absorbed into the coordinate functions. When  $D < 0$ , the conformal factor in (B2) is valid only in the elsewhere region of the light cone  $L(T_0)$ , while with  $D > 0$ , it is valid only within the causal regions of  $L(T_0)$  (see left panel of Fig 2). These two subfamilies are complementary in that together they span the entire space time. Note that, under this transformation, all the points on the light cone  $L(T_0)$  map to infinity — more specifically, to the boundary of the conformal completion of Minkowski space. Hence, choosing a specific coordinate transformation first involves choosing a specific light cone  $L(T_0)$ , and then choosing to span either its causal region or its elsewhere region. The  $\pm$  sign in front of the time coordinate corresponds to the time reversal symmetry intrinsic to the metric. An appropriate sign can be chosen by requiring that  $T'$  is a monotonically increasing function of  $T$ . Thus, when  $D > 0$ , we shall choose the - sign, and Future and Past map into each other. When  $D < 0$ , we shall choose the + sign, and map spacelike Elsewhere into itself.

We now consider two different light cones  $L(T_{01})$  and  $L(T_{02})$ . Correspondingly, we choose two transformations  $\{H_1, T', R'\}$  and  $\{H_2, \tilde{T}, \tilde{R}\}$ . Clearly, in the region of overlap of these two transformations, the new coordinate functions can be expressed in terms of one another. That is, we can express  $\{H_2, \tilde{T}, \tilde{R}\}$  as a function of  $\{T', R'\}$ .

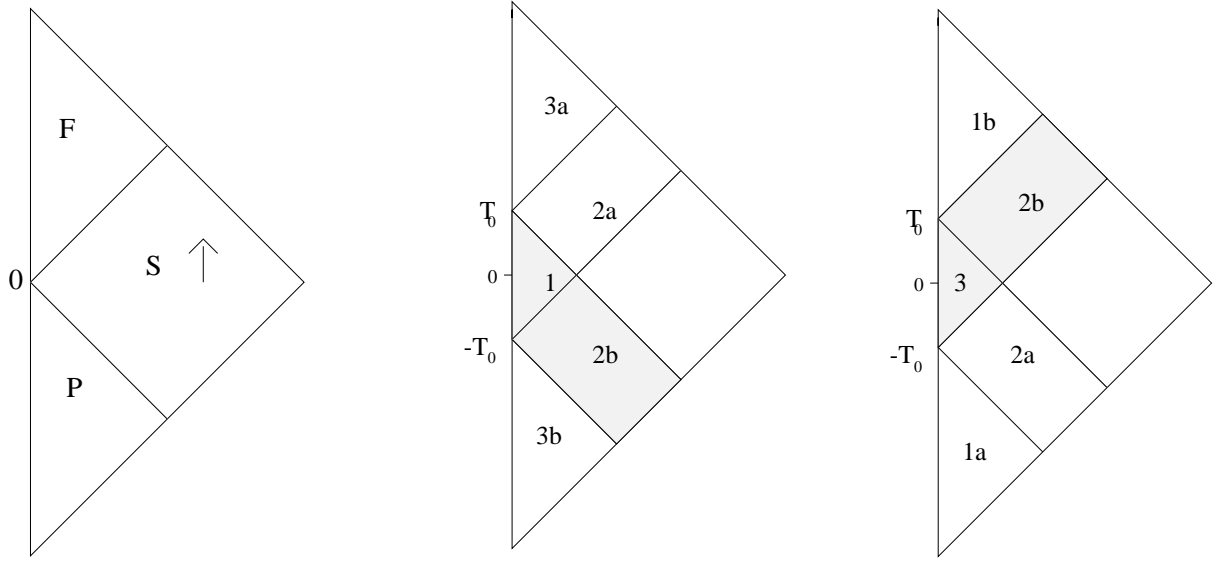


FIG. 2: At the left, Minkowski space is shown, with Future, Past and Spacelike regions (relative to the origin) labeled, and time-orientation indicated. In the middle, the light cones above/below  $\mp T_0$  map to the light cones below/above  $\pm T_0$  on the right. Regions 2a and 2b in the middle map to regions with the same name on the right, and neighboring regions map accordingly. The type of map which takes 1a on the right directly to 1b on the right is characterized by (B4).

The generality of (B2) ensures that this new transformation is of the same form as (B2). We find the transformation explicitly and show its conformance with (B2).

We choose the two coordinate systems  $(T', R')$  and  $(\tilde{T}, \tilde{R})$  to span the causal region within the respective light cones:

$$\left\{ \begin{array}{l} H_1 = D_1 [(T - T_{01})^2 - R^2], \\ T' - T'_{01} = -(T - T_{01})/H_1, \\ R' = R/H_1, \end{array} \right\} \text{ and } \left\{ \begin{array}{l} H_2 = D_2 [(T - T_{02})^2 - R^2], \\ \tilde{T} - \tilde{T}_{02} = -(T - T_{02})/H_2, \\ \tilde{R} = R/H_2. \end{array} \right\} \quad (\text{B3})$$

In the region of overlap of the two coordinate systems (region 1 in the middle panel of Fig 2)  $\tilde{T}$  and  $\tilde{R}$  can be expressed in terms of  $T', R'$  (regions 1a and 1b in the panel on the right of Fig 2). Note that time orientation has been preserved:

$$\left\{ \begin{array}{l} H = \tilde{D} [(T' - T'_0)^2 - R'^2], \\ \tilde{T} - \tilde{T}_0 = -(T' - T'_0)/H, \\ \tilde{R} = R'/H, \end{array} \right\} \text{ where } \left\{ \begin{array}{l} \tilde{D} = D_1 D_2 (T_{01} - T_{02})^2, \\ T'_0 = T'_{01} + 1/[D_1 (T_{01} - T_{02})], \\ \tilde{T}_0 = \tilde{T}_{02} - 1/[D_2 (T_{01} - T_{02})]. \end{array} \right\} \quad (\text{B4})$$

This is clearly seen to be of the same form as (B2). The properties of the conformal factor  $H(T, R)$  demonstrated in this appendix are used in section VIA to elucidate the properties of the conformal factors  $\Omega(t, r)$  obtained in (25) and (A18).

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